Math 255A Lecture 12 Notes

Daniel Raban

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1 Unbounded Fourier Coefficients and Bilinear Maps

1.1 Unbounded partial sums of Fourier coefficients

Last time we introduced an application of the Banach-Steinhaus theorem. Let $S_n(f,0) = \sum_{-N}^{N} c_n(F)$, where $c_n(f)$ is the n-th Fourier coefficient of f.

Proposition 1.1. *There exists a 2π-periodic $f \in C(\mathbb{R})$ such that the sequence $(S_N(f,0))_{N=1}^{\infty}$ is unbounded.*

Proof.

$$S_n(f,0) = \sum_{n=-N}^{N} c_n(f) = \int_{-\pi}^{\pi} D_N(x) f(x)\, dx,$$

where $D(x) = \sum_{-N}^{N} e^{inx}$ is the Dirichlet kernel. We have

$$D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

If the claim does not hold, we have that $(S_n(f,0))$ is bounded for all $f \in B$, the Banach space of continuous 2π-periodic functions with $\|f\|_B = \sup_{[-\pi,\pi]} |f|$. By the Banach-Steinhaus theorem, there exists $C > 0$ such that $|S_N(f,0)| \leq C\|f\|_B$ for all $f \in B$ and $N \in \mathbb{N}^+$. So

$$\left| \int_{-\pi}^{\pi} D_N(x) f(x)\, dx \right| \leq \|f\|_B \implies \|D_N\|_{L^1(-\pi,\pi)} \leq 1.$$

On the other hand,

$$\begin{aligned} \|D_N\|_{L^1(-\pi,\pi)} &= \frac{2}{2\pi} \int_0^{\pi} \frac{|\sin((N+1/2)x)|}{\sin(x/2)} \\ &\geq \frac{4}{2\pi} \int_0^{\pi} \frac{|\sin((N+1/2)x)|}{x}\, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^{(N+1/2)\pi} \frac{|\sin(x)|}{x}\,dx \\
&\geq \frac{2}{\pi}\sum_{n=1}^{N-1}\int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{x}\,dx \\
&\geq \frac{4}{\pi^2}\sum_{n=2}^{N}\frac{1}{n} \\
&= \frac{4}{\pi^2}\log(N) + O(1)
\end{aligned}
$$

as $N \to \infty$. If follows that the set of all $f \in B$ such that $(S_N(f,0))_{N=1}^{\infty}$ is bounded is of the first category. By translation invariance, we get the same statement for $(S_N(f,x))_{N=1}^{\infty}$ for each fixed $x \in \mathbb{R}$. Taking the union over all $x \in \mathbb{Q}$, we get a set of the first category such that if f is in the complement, then $(S_n(f,x))_{N=1}^{\infty}$ is unbounded for all $x \in \mathbb{Q}$. $\square$

Remark 1.1. Notice that for all $f \in B$, we have $S_N(f,x) = o(\log(N))$ uniformly in x, as $N \to \infty$. This follows as $\|D\|_{L^1} = O(\log(N))$ and $S_N(f,x) = O(1)$ for 2π-periodic $f \in C^1(\mathbb{R})$ (dense in B).

1.2 Bilinear maps

Let E, F, G be locally convex spaces, and let $B : E \times F \to G$ be bilinear.

Proposition 1.2. *Assume that B is continuous at $0 \in E \times F$. Then B is continuous.*

Proof. Let U_G be a neighborhood of $0 \in G$, and let U_E, U_F be neighborhoods of 0 in E, F such that if $x \in U_E$ and $y \in U_F$, then $B(x,y) \in U_G$. Write $B(x+x_0, y+y_0) = B(x,y) + B(x,y_0) + B(x_0,y) + B(x_0,y_0)$. As U_E, U_F are absorbing, let $\varepsilon > 0$ be such that $\varepsilon x_0 \in U_E$ and $\varepsilon y \in U_G$. Then $B(x,y_0) = B(x/\varepsilon, \varepsilon y_0) \in U_G$ if $x/\varepsilon \in U_E$. Similarly, $B(x_0,y) \in U_G$ if $y/\varepsilon \in U_F$. When $x \in U_E \cap \varepsilon U_E$ and $y \in U_F \cap \varepsilon U_F$, $B(x+x_0, y+y_0) - B(x_0,y_0) \in U_G + U_G + U_G$. $\square$

We have that $B : E \times F \to G$ is continuous iff for every continuous seminorm p_G on G, there exist continuous seminorms p_E on E and p_F on F such that

$$p_G(B(x,y)) \leq p_E(x)p_F(y)$$

for all $x \in E$ and $y \in F$.

Definition 1.1. We say that a bilinear form B is **separately continuous** if the linear forms $x \mapsto B(x,y)$ for fixed y and $y \mapsto B(x,y)$ for fixed x are continuous.

Theorem 1.1. *Let E be locally convex and metrizable, F a Fréchet space, and G a locally convex space. If the bilinear form $B : E \times F \to G$ is separately continuous, then B is continuous.*

Proof. Let U be a an open, convex, symmetric neighborhood of $0 \in G$. Let $V_1 \supseteq V_2 \supseteq \cdots$ be a fundamental system of neighborhoods of 0 in E. Let

$$A_j = \{y \in F : B(x,y) \in \overline{U}\ \forall x \in V_j\} = \bigcap_{x \in V_j} B^{-1}(x,\cdot)(\overline{U})$$

As $y \mapsto B(x,y)$ is continuous, A_j is closed. It is also convex and symmetric. For any $y \in F$, $x \mapsto B(x,y)$ is continuous, so there exists j such that $x \in V_j \implies B(x,y) \in \overline{U}$. In other words, $\bigcup_{j=1}^{\infty} A_j = F$. By the open mapping theorem, there exists some j such that A_j has an interior point. Arguing as in the proof of the Banach-Steinhaus theorem, we get that 0 is an interior point of A_j; i.e. there exists a neighborhood N of $0 \in F$ such that if $y \in N$ and $x \in V_j$, $B(x,y) \in U$. Thus, B is continuous at 0 and hence continuous. □

Remark 1.2. It suffices to have a locally convex topology on E defined by countably many seminorms (no Hausdorff property is needed).